«بسمه تعالی»

[illegible]

① $\displaystyle\int \frac{\overset{\overset{2\sin x\cos x}{\uparrow}}{\sin 2x}}{\sqrt{2+\sin x}}\,dx \qquad \sqrt{2+\sin x} = u \longrightarrow \sin x = u^2-2 \longrightarrow \cos x\,dx = 2u\,du$

$$= \int \frac{2(u^2-2)\,2u\,du}{u^2} = 4\int \frac{(u^2-2)}{u}\,du = 4\int u\,du - 8\int \frac{du}{u}$$

$$= 4\,\frac{u^2}{2} - 8\ln|u| + C$$

$$= 2\left(\sqrt{2+\sin x}\right)^2 - 8\ln\left|\sqrt{2+\sin x}\right| + C$$

② $\displaystyle\int \frac{x-1}{\sqrt{1+4x}}\,dx \qquad \sqrt{1+4x} = u \longrightarrow 4x = u^2-1, \quad 4\,dx = 2u\,du$

$$= \int \frac{\left[\left(\frac{u^2-1}{4}\right)-1\right]\left(\frac{u}{2}\,du\right)}{u} = \frac{1}{8}\int (u^2-5)\,du = \frac{1}{8}\left(\frac{u^3}{3} - 5u\right) + C$$

$$= \frac{1}{8}\left(\frac{(\sqrt{1+4x})^3}{3} - 5(\sqrt{1+4x})\right) + C$$

③ $\displaystyle\int x\ln x\,dx \qquad \begin{cases} \ln x = u \longrightarrow \frac{dx}{x} = du \\ x\,dx = dv \longrightarrow v = \frac{x^2}{2} \end{cases}$

$$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{dx}{x} = \frac{x^2}{2}\ln x - \frac{1}{2}\,\frac{x^2}{2} + C$$

④ $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx \qquad x = 3\sin\theta \ (*) \longrightarrow dx = 3\cos\theta\,d\theta$

$$= \int \frac{(3\sin\theta)^3\,(3\cos\theta\,d\theta)}{\underbrace{\sqrt{9-9\sin^2\theta}}_{3\cos\theta}} = 27\int \sin^3\theta\,d\theta = 27\int \sin^2\theta\cdot\sin\theta\,d\theta$$

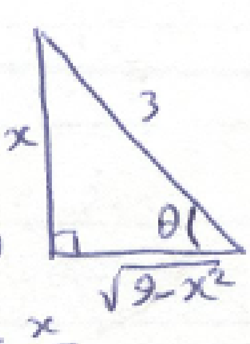
$$= 27\int (1-\cos^2\theta)\sin\theta\,d\theta \qquad u = \cos\theta, \quad du = -\sin\theta\,d\theta$$

$$= 27\int (1-u^2)(-du) = -27\int (1-u^2)\,du = -27\left(u - \frac{u^3}{3}\right) + C$$

$$= -27\left(\cos\theta - \frac{\cos^3\theta}{3}\right) + C$$

$$= -27\left(\frac{\sqrt{9-x^2}}{3} - \frac{(\sqrt{9-x^2})^3}{3}\right) + C$$

(*)



$x = 3\sin\theta$

$\sin\theta = \frac{x}{3}$

۵ $\displaystyle \int \frac{x+1}{x^2(x-1)}\,dx$

کسر گویا

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax(x-1)+B(x-1)+Cx^2}{x^2(x-1)}$$

$$\begin{cases} A+C=0 \longrightarrow C=2 \\ -A+B=1 \longrightarrow A=-2 \\ -B=1 \longrightarrow B=-1 \end{cases} \Rightarrow \int \frac{-2}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{2}{x+1}\,dx$$

$$= -2\ln|x| - \left(\frac{-1}{x}\right) + 2\ln|x+1| + C$$

۶ $\displaystyle \int tg^2 x\,dx = \int \left((1+tg^2x)-1\right)dx = tg\,x - x + C$

۷ $\displaystyle \int x^2 \sin(2x+1)\,dx$ $\qquad \begin{cases} x^2 = u \to 2x\,dx = du \\ \sin(2x+1)\,dx = dv \to v = -\frac{1}{2}\cos(2x+1) \end{cases}$

$$= -\frac{1}{2}x^2\cos(2x+1) - \int \left(-\frac{1}{\cancel{2}}\cos(2x+1)\right)(\cancel{2}x\,dx)$$

$x = u \to d$

$\cos(2x+1)\,dx = dv$

$\downarrow$

$v = \frac{1}{2}(\sin$

$$= -\frac{1}{2}x^2\cos(2x+1) + \left[\frac{1}{2}x\sin(2x+1) - \int \frac{1}{2}\sin(2x+1)\,dx\right]$$

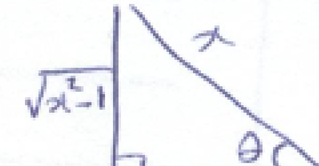
$$= -\frac{1}{2}x^2\cos(2x+1) + \frac{1}{2}x\sin(2x+1) + \frac{1}{4}\cos(2x+1) + C$$

۸ $\displaystyle \int \frac{4x^2+x-2}{\underbrace{x^3-x^2}_{x^2(x-1)}}\,dx$ $\qquad \displaystyle \frac{4x^2+x-2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

$$\Rightarrow \begin{cases} A+C=4 \to C=3 \\ -A+B=1 \to A=1 \\ -B=-2 \to B=2 \end{cases} \qquad \int \frac{dx}{x} + \int \frac{2}{x^2}\,dx + \int \frac{3}{x-1}\,dx$$

$$= \ln|x| - \frac{2}{x} + 3\ln|x-1| + C$$

۹ $\displaystyle \int \frac{dx}{x^2\sqrt{x^2-1}}$ $\qquad x = \sec\theta \qquad dx = \sec\theta\, tg\theta\, d\theta$



$$= \int \frac{\sec\theta\, tg\theta\, d\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}} = \int \frac{\sec\theta\, tg\theta\, d\theta}{\sec^2\theta \cdot tg\theta} = \int \frac{1}{\sec\theta}\,d\theta$$

$$= \int \frac{1}{\frac{1}{\cos\theta}}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C = \frac{\sqrt{x^2-1}}{x} + C$$

(۱۰) $\int (\ln x)^2\, dx \qquad (\ln x)^2 = u \rightarrow 2\ln x \cdot \frac{dx}{x} = du$

$\qquad\qquad\qquad\qquad dx = dv \rightarrow v = x$

$= x(\ln x)^2 - \int x \cdot (2\ln x)\frac{dx}{x} \qquad \ln x = u \rightarrow \frac{dx}{x} = du$

$\qquad\qquad\qquad\qquad dx = dv \rightarrow x = v$

$= x(\ln x)^2 - \left[x\ln x - \int x \frac{dx}{x} \right] = x(\ln x)^2 - x\ln x + x + C$

(۱۱) $\int \frac{dx}{\sqrt{x - x^2}} = \int \frac{dx}{\sqrt{-(x^2 - x)}} = \int \frac{dx}{\sqrt{\frac{1}{4} - (x - \frac{1}{2})^2}}$ $\quad \boxed{u^2 = (x - \tfrac{1}{2})^2 \\ a^2 = \tfrac{1}{4}}$

$dx = \frac{1}{2}\cos\theta\, d\theta \leftarrow x - \frac{1}{2} = \frac{1}{2}\sin\theta \leftarrow$

$2x-1$

$= \int \frac{\frac{1}{2}\cos\theta\, d\theta}{\sqrt{\frac{1}{4} - (\frac{1}{2}\sin\theta)^2}} = \int \frac{\frac{1}{2}\cos\theta\, d\theta}{\frac{1}{2}\cos\theta} = \int d\theta = \theta + C = \sin^{-1}(2x$

(۱۲) $\int \frac{dx}{5 + 7\cos x + \sin x} = \int \frac{dx}{5 + 7\left(\frac{1 - tg^2\frac{x}{2}}{1 + tg^2\frac{x}{2}}\right) + \frac{2\, tg\frac{x}{2}}{1 + tg^2\frac{x}{2}}}$

$= \int \frac{(1 + tg^2\frac{x}{2})\, dx}{12 - 2tg^2\frac{x}{2} + 2tg\frac{x}{2}} \qquad \begin{cases} tg\frac{x}{2} = u \\ \frac{1}{2}(1 + tg^2\frac{x}{2})\, dx = du \end{cases}$

تشابه

$= \int \frac{2\, du}{12 - 2u^2 + 2u} = -\int \frac{du}{\underbrace{u^2 - u - 6}_{(u+2)(u-3)}} = -\left[\frac{1}{5}\int \frac{du}{u+2} - \frac{1}{5}\int \right.$

$= -\frac{1}{5}\ln|u+2| + \frac{1}{5}\ln$

$= -\frac{1}{5}\ln\left|tg(\frac{x}{2}) + 2\right| + \frac{1}{5}\ln\left|\right.$

(۱۳) $\int \frac{(x+1)}{x^2\sqrt{1+x^2}}\, dx \qquad x = tg\theta \qquad dx = \sec^2\theta\, d\theta$

$= \int \frac{(1 + tg\theta)\sec^2\theta\, d\theta}{tg^2\theta(\sqrt{1 + tg^2\theta})} = \int \frac{(1 + tg\theta)\sec^2\theta}{tg^2\theta(\sec\theta)}\, d\theta = \int \frac{\sec\theta + \sec\theta\, tg}{tg^2\theta}$

$= \int \frac{\cos\theta}{\sin^2\theta}\, d\theta + \int \csc\theta\, d\theta = \int \frac{du}{u^2} + \int \csc\theta\, d\theta$

$\sin\theta = u \rightarrow \cos\theta\, d\theta = du \qquad = -\frac{1}{u} + \int \csc\theta\, d\theta$

مشابه $\int \sec\theta\, d\theta$ با [illegible]

۱۴) $\displaystyle\int e^{2x}\ln(e^x+1)\,dx$ $\qquad \ln(e^x+1)=u \longrightarrow \dfrac{e^x}{e^x+1}\,dx = du$

$e^{2x}\,dx = dv \longrightarrow v = \frac{1}{2}e^{2x}$

$$= \frac{1}{2}e^{2x}\ln(e^x+1) - \int \frac{1}{2}e^{2x}\cdot\frac{e^x}{e^x+1}\,dx$$

$$= \frac{1}{2}e^{2x}\ln(e^x+1) - (*)$$

حل $\displaystyle\int \frac{1}{2}\,\frac{\overset{e^{3x}}{\overbrace{e^{2x}\cdot e^{x}}}}{e^x+1}\,dx \Rightarrow \quad e^x+1=t \longrightarrow e^x\,dx = dt$

$\qquad dx = \dfrac{dt}{e^x} = \dfrac{dt}{t-1}$

$$= \frac{1}{2}\int \frac{(t-1)^{\cancel{3}}}{t}\cdot\frac{dt}{\cancel{(t-1)}} = \frac{1}{2}\int \frac{(t-1)^2\,dt}{t} = \frac{1}{2}\int \frac{(t^2-2t+1)}{t}\,dt$$

$$= \frac{1}{2}\int \left(t-2+\frac{1}{t}\right)dt = \frac{1}{2}\left[\frac{t^2}{2}-2t+\ln|t|\right]+C$$

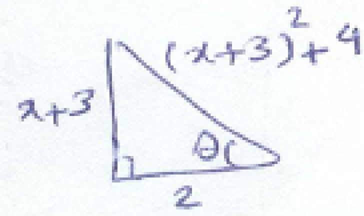
$$= \frac{1}{2}\left[\frac{(e^x+1)^2}{2}-2(e^x+1)+\ln|e^x+1|\right]+C$$

۱۵) $$\int \frac{dx}{\sqrt{(x^2+6x+13)^3}} = \int \frac{dx}{\sqrt{(x^2+6x+9)+4)^3}}$$

$$= \int \frac{dx}{\sqrt{((x+3)^2+4)^3}}$$

$x+3 = 2\,tg\,\theta$

$dx = 2\sec^2\theta\,d\theta$



$$= \int \frac{2\sec^2\theta\,d\theta}{\sqrt{(\underbrace{4tg^2\theta+4}_{\underbrace{4(1+tg^2\theta)}_{\sec^2\theta}})^3}} = \int \frac{2\sec^2\theta\,d\theta}{8\sec^3\theta} = \frac{1}{8}\int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{8}\int \cos\theta\,d\theta$$

$$= \frac{1}{8}\sin\theta + C$$

$$= \frac{1}{8}\left(\frac{(x+3)}{(x+3)^2+4}\right)+C$$

۱۶) $\displaystyle\int 4x^3\,tg^{-1}x\,dx$

$tg^{-1}x = u \longrightarrow du = \dfrac{dx}{1+x^2}$

$4x^3\,dx = dv \longrightarrow v = x^4$

$$= x^4\,tg^{-1}x - \int \frac{x^4}{1+x^2}\,dx$$

$$= x^4\,tg^{-1}x - \left[\int\left((x^2-1)+\frac{1}{x^2+1}\right)dx\right]$$

$$= x^4\,tg^{-1}x - \frac{x^3}{3} + x - tg^{-1}x + C$$

$$\begin{array}{r|l} x^4 & x^2+1 \\ \hline \mp x^4 \mp x^2 & x^2-1 \\ -x^2 & \\ \pm x^2 \pm 1 & \\ \hline 1 & \end{array}$$

۱۷

$$\int \left(\frac{\csc\theta}{\csc\theta - \sin\theta} \right) d\theta$$

$$= \int \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \sin\theta}\, d\theta = \int \frac{\frac{1}{\sin\theta}}{\frac{1-\sin^2\theta}{\sin\theta}}\, d\theta = \int \frac{۱}{\cos^2\theta}\, d\theta$$

$$= \int \sec^2\theta \, d\theta = tg\,\theta + C$$

۱۸

$$\int \frac{dx}{\sqrt{x+1} + \sqrt{x-1}}$$

$$\int \frac{1}{\sqrt{x+1}+\sqrt{x-1}} \times \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\, dx = \int \frac{\sqrt{x+1}-\sqrt{x-1}}{\underbrace{(x+1)-(x-1)}_{2}}\, dx$$

$$\frac{1}{2}\int \left(\sqrt{x+1} - \sqrt{x-1}\right) dx = \frac{1}{2}\left[\frac{(x+1)^{3/2}}{3/2} - \frac{(x-1)^{3/2}}{3/2} \right] + C$$

۱۹

$$\int \frac{1-\cos x}{1+\cos x}\, dx \qquad \cos x = \frac{1-tg^2 \frac{x}{2}}{1+tg^2\frac{x}{2}}$$

$$= \int \frac{1 - \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}}{1+\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}}\, dx = \int \frac{\cancel{2}\, tg^2\frac{x}{2}}{\cancel{2}}\, dx = \int \left(tg^2\frac{x}{2} + 1 - 1\right)$$

$$= 2\, tg\frac{x}{2} - x + C$$

۲۰

$$\int \sin^2\left(\frac{1}{2}\cos^{-1}x\right) dx \qquad \sin^2 u = \frac{1-\cos 2u}{2}$$

$$= \int \frac{1-\cos 2\left(\frac{1}{2}\cos^{-1}x\right)}{2}\, dx = \int \frac{1-\cos\left(\cos^{-1}x\right)}{2}\, dx$$

$$= \int \frac{1-x}{2}\, dx = \frac{1}{2}x - \frac{1}{2}\frac{x^2}{2} + C$$